UTAP-585
RESCEU-84/07

Constraints on Wrapped DBI Inflation in a Warped Throat

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(Dated: February 4, 2008)

We derive constraints on the tensor to scalar ratio and on the background charge of the warped throat for DBI inflation driven by D5- and D7-branes wrapped over cycles of the throat. It is shown that the background charge well beyond the known maximal value is required in most cases for DBI inflation to generate cosmological observables compatible with the WMAP3 data. Most of the results derived in this paper are insensitive to the details of the inflaton potential, and could be applied to generic warped throats.

I. INTRODUCTION

Numerous attempts have been made to explain inflation from string theory. Among them, brane inflation in a flux compactified warped throat [1] is one of the most promising ideas. However, it was soon realized that this model had a serious problem in realizing slow-roll without the help of fine tuning. Ever since, many interesting ideas have been put forward to overcome this so-called η problem ([2, 3, 4], etc.).

On the other hand, an alternative approach to the η problem which gives up slow-roll was suggested in the idea of DBI inflation [5, 6]. In this model, a relativistic motion of a D3-brane is considered. Nevertheless, the potential energy dominates over the kinetic energy since the latter is suppressed due to the warping of the throat, which leads to an accelerated expansion of the universe.

Besides being a remedy for the η problem, DBI inflation has a further exciting property which gathered attention. It has a possibility of producing signals in the temperature fluctuation in the cosmic microwave background radiation within the observable range of future detectors. Especially, it is expected to produce large non-Gaussianity. These features can be attributed to DBI inflation being a kind of k-inflation with general speed of sound [7, 8].

However, recently it has been pointed out that DBI inflation driven by a mobile D3-brane might contradict the current WMAP3 data [9, 10, 11, 12]. One of the essential points is that one can derive an universal lower bound on the tensor to scalar ratio r under the assumption that the non-Gaussianity is large. Another, more microscopic, point is that r is related to the change of the inflaton $\Delta\phi$ by the so-called Lyth bound [13] (see (30) below). Therefore, the lower bound of r requires large $\Delta\phi$ over observable scales. However, since the inflaton field represents the radial position of the brane, $\Delta\phi$ is restricted by the size of the extra dimensions. We shall review more details of the argument in Section IV but the essence is, as mentioned above, the universal lower bound of r implied by the large non-Gaussianity and the relation between $\Delta\phi$ and the size of the extra dimensions.

In this paper we consider a simple extension of the DBI inflation model in which the relation between $\Delta\phi$ and the size of the extra dimensions is modified. We shall consider higher dimensional D-branes wrapped over cycles of the throat, instead of a simple D3-brane. The volume of the cycles appears as an overall factor of the kinetic term of the inflaton. Thus, properly normalizing the definition of the inflaton field, the relation between $\Delta\phi$ and the size of the extra dimensions should be modified. The larger the volume of the cycles, the larger $\Delta\phi$ for the same size of extra dimensions. Therefore, it is rather natural to expect that higher dimensional, wrapped D-branes is one of the simplest scenarios to bypass the above mentioned problem of DBI inflation.

For this reason, in this paper, we focus on a D5- or D7-brane moving with a relativistic speed towards the tip of a warped throat, hoping to find a model which generates large non-Gaussianity and is consistent with WMAP3 data. Unfortunately, contrary to the hope, in many cases we find difficulties. We find that DBI inflation with a D5-brane requires a large Euler number of a Calabi–Yau four-fold, exceeding the known maximal value $\chi = 1820448$ [14]. However, we also show that a D7-brane may be able to excite DBI inflation provided that the observed CMB scale is

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produced when the D7-brane is in the region of the throat where the contribution of the NS-NS 2-form potential B_2 to the action is substantial.

The outline of the paper is as follows: In Section II we lay out the basic setup. The expressions for the cosmological observables in DBI inflation are reviewed in Section III. In Section IV we review the constraints on gravitational waves derived by Lidsey and Huston [11], and then extend the discussion to higher dimensional D-branes in Section V. We present more stringent bounds focusing on background charge in Section VI, and we conclude in Section VII. As an example of a flux compactified warped throat, the Klebanov–Strassler solution [15, 16] is introduced in Appendix A. The behavior of the inflaton near the tip of the throat is discussed in Appendix B. In Appendix C, we discuss the number of e-foldings produced in the region where B_2 can be neglected. Also, a brief discussion on $\det(G_{kl} - B_{kl})$ is given in Appendix D.

II. THE BASIC SETUP

We assume a warped throat background with its moduli stabilized based on flux compactification of type IIB string theory. The generic 10-dimensional metric takes the following form:

$$ds^2 = h^2(\rho)\,\eta_{\mu\nu}\,dx^\mu dx^\nu + h^{-2}(\rho)\left(d\rho^2 + \rho^2\, g^{(5)}_{mn}\,dx^m dx^n\right), \tag{1}$$

where x^μ $(\mu = 0,\ 1,\ 2,\ 3)$ are the external 4-dimensional coordinates, ρ is the radial coordinate which decreases as it approaches the tip of the throat, and x^m $(m = 5,\ 6,\ 7,\ 8,\ 9)$ are the internal 5-dimensional angular coordinates.

In this paper, we investigate conditions that can be imposed upon *DBI inflation caused by a D-brane moving with relativistic speed towards the tip of a warped throat.* Since type IIB string theory contains stable Dp-branes with p odd, we focus on D3-, D5-, and D7-branes and collectively refer to them as D(3+2n)-branes ($n = 0,\ 1,\ 2$). We assume that a D(3+2n)-brane stretches out along the external space and also wraps a $2n$-cycle in the angular directions of the internal space.

DBI inflation is motivated by the Dirac–Born–Infeld (DBI) action, which takes the following form for a D(3+2n)-brane,

$$S = -T_{3+2n}\int d^{4+2n}\xi\, e^{-\Phi}\sqrt{-\det(G_{AB} - B_{AB})}, \tag{2}$$

where G_{AB} is the induced metric on the D-brane world-volume, B_{AB} is the pull-back of the NS-NS 2-form flux B_2, Φ is the dilaton, and $T_{3+2n} = 1/(2\pi)^{3+2n} g_s \alpha'^{n+2}$ is the brane tension. We assume that the dilaton is stabilized to a constant value and is set to 0 hereafter.

We take the first four brane coordinates ξ^α to coincide with x^α $(\alpha = 0, \cdots, 3)$ and that the angular position of the brane in the internal space are functions of the remaining $2n$ brane coordinates $x^m = x^m(\xi^l)$. Then, by further assuming that the D-brane radial position ρ depends only on ξ^α, the induced metric is

$$G_{AB}\,d\xi^A\ d\xi^B = (h^2\eta_{\alpha\beta} + h^{-2}\partial_\alpha\rho\,\partial_\beta\rho)d\xi^\alpha\,d\xi^\beta + G_{kl}\,d\xi^k\,d\xi^l, \tag{3}$$

where

$$G_{kl} = h^{-2}\rho^2 g^{(5)}_{mn}\frac{\partial x^m}{\partial \xi^k}\frac{\partial x^n}{\partial \xi^l}. \tag{4}$$

For B_2, we assume it to have a logarithmic radial dependence and have components only along the angular directions. This is a well motivated property for B_2 in order for the dual description to reproduce a logarithmic flow of couplings found in field theory [17, 18]. However, we postpone the explicit form of B_2 until Section VI and here we merely state that the legs of B_2 is along the angular directions of the internal space:

$$B_2 = \frac{1}{2}b_{mn}dx^m \wedge dx^n, \tag{5}$$

where b_{mn} is antisymmetric in its indices.

Then the DBI action for a D(3+2n)-brane takes the following form,

$$S = -T_{3+2n}\int d^4\xi\, h^4\sqrt{1 + h^{-4}\eta^{\alpha\beta}\partial_\alpha\rho\partial_\beta\rho}\int d^{2n}\xi\,\sqrt{\det(G_{kl} - B_{kl})}, \tag{6}$$

where

$$B_{kl} = b_{mn} \frac{\partial x^m}{\partial \xi^k} \frac{\partial x^n}{\partial \xi^l}. \tag{7}$$

Assuming that the physical energy scale associated with the 4-dimensional universe is much lower than the energy scale of moduli stabilization, we promote the 4-dimensional flat metric $\eta_{\mu\nu} dx^\mu dx^\nu$ to a curved metric $g^{(4)}_{\mu\nu} dx^\mu dx^\nu$:

$$S = -T_{3+2n} \int d^4\xi \sqrt{-g^{(4)}} h^4 \sqrt{1 + h^{-4} g^{(4)\alpha\beta} \partial_\alpha \rho \partial_\beta \rho} \int d^{2n}\xi \sqrt{\det(G_{kl} - B_{kl})}. \tag{8}$$

Introducing a new variable and a function

$$d\phi \equiv T_{3+2n}^{1/2} \left\{ \int d^{2n}\xi \sqrt{\det(G_{kl} - B_{kl})} \right\}^{1/2} d\rho, \tag{9}$$

$$T(\phi) \equiv T_{3+2n} h^4 \int d^{2n}\xi \sqrt{\det(G_{kl} - B_{kl})}, \tag{10}$$

the action turns into a simple form

$$S = -\int d^4\xi \sqrt{-g^{(4)}}\, T(\phi) \sqrt{1 + g^{(4)\alpha\beta} \partial_\alpha \phi \partial_\beta \phi / T(\phi)}. \tag{11}$$

It should be noted that the differences in results we derive in this paper between D3-branes and higher dimensional wrapped D-branes originate in the normalization factor of the inflaton in (9).

Further adding a Chern–Simons term and the inflaton potential to the DBI action coupled to gravity, the full inflaton action takes the familiar form

$$S = \int d^4\xi \sqrt{-g^{(4)}} \left[\frac{M_p^2}{2} \mathcal{R} - \left\{ T(\phi) \sqrt{1 + g^{(4)\alpha\beta} \partial_\alpha \phi \partial_\beta \phi / T(\phi)} - T(\phi) + V(\phi) \right\} \right], \tag{12}$$

where M_p is the reduced Planck mass.

Hereafter, we assume that the metric $g^{(4)}_{\alpha\beta}$ is the physical 4-dimensional metric which directly couples to matter fields on the Standard Model brane. In general, from the viewpoint of the 4-dimensional effective field theory there is no symmetry argument to prohibit the possibility that the Standard Model fields may be coupled to a conformally transformed metric $\Omega^2 g^{(4)}_{\alpha\beta}$ rather than $g^{(4)}_{\alpha\beta}$ itself, where Ω is a function of the inflaton. However, from the higher-dimensional point of view, if the Standard Model brane is geometrically separated from branes responsible for inflation then such a coupling should be highly suppressed. This is the situation expected in multi-throat scenarios, where the Standard Model brane is in a different throat from the inflationary throat. (Further motivations for considering multi-throat scenarios are reviewed in [19, 20, 21].) To be more precise, the induced metric on the Standard Model brane is $g^{(4)}_{\alpha\beta}$ only up to a conformal factor. However, this conformal factor is essentially independent of the inflaton for the reason explained above. Thus, this conformal factor can be considered as a constant at low energy as far as all moduli, including volume and shape of extra dimensions and the position of the Standard Model brane, are properly stabilized. By rescaling the unit of reference, one can set the constant conformal factor to 1. (It should be noted that this rescaling does not affect the values of the dimensionless cosmological observables, such as the ones we introduce in the next section. However, it does change the local string scale on the Standard Model brane so that a large hierarchy may be generated a la Randall and Sundrum [22].) For these reasons, throughout the present paper we suppose that matter fields on the Standard Model brane are directly coupled to the metric $g^{(4)}_{\alpha\beta}$.

Recently there have been attempts to study the inflaton potential in detail for D3-branes [23, 24, 25]. Nevertheless, with our present understanding of string theory, it is fair to say that the form of the potential $V(\phi)$ is not well under theoretical control, let alone the potential for wrapped D-branes. Therefore, in this paper we seek constraints on DBI inflation without specifying the form of the potential. In other words, the results of this paper depend only on the kinetic term of the inflaton action, and are insensitive to the Chern–Simons term and the potential term. We leave the potential arbitrary and focus only on the DBI part of the action.

III. COSMOLOGICAL OBSERVABLES

Cosmological observables such as the spectrum of density and tensor perturbations generated by the action (12) have been studied in [6, 8, 26, 27]. We now briefly review the expressions of cosmological observables for DBI inflation. Due

to the nontrivial form of the kinetic term in the action (12), DBI inflation can be interpreted as a kind of k-inflation with a general speed of sound [7, 8].

By taking the functional derivative of the action with respect to the 4-dimensional metric $g^{(4)}_{\alpha\beta}$, we obtain the following expression for the stress-energy tensor

$$T_{\alpha\beta} = \gamma \partial_\alpha \phi \partial_\beta \phi - g^{(4)}_{\alpha\beta} \left[T(\phi)(\gamma^{-1} - 1) + V(\phi) \right], \tag{13}$$

where γ is an analog of a Lorentz factor in special relativity

$$\gamma \equiv \frac{1}{\sqrt{1 + g^{(4)\alpha\beta} \partial_\alpha \phi \partial_\beta \phi / T(\phi)}}. \tag{14}$$

For the FRW background with a homogeneous ϕ, the pressure p and the energy density ρ take the following form

$$p = T(\phi)(1 - \gamma^{-1}) - V(\phi), \quad \rho = T(\phi)(\gamma - 1) + V(\phi), \tag{15}$$

and γ is

$$\gamma = \frac{1}{\sqrt{1 - \dot{\phi}^2 / T(\phi)}}. \tag{16}$$

The speed of sound relevant to inhomogeneous perturbations is given by

$$c_s = \frac{1}{\gamma}. \tag{17}$$

We define the following parameters (often called DBI parameters) in analogy with the usual slow-roll parameters

$$\tilde{\epsilon} \equiv \frac{2M_p^2}{\gamma} \left(\frac{H'}{H} \right)^2, \tag{18}$$

$$\tilde{\eta} \equiv \frac{2M_p^2 H''}{\gamma H}, \tag{19}$$

$$\tilde{s} \equiv \frac{2M_p^2 \gamma' H'}{\gamma^2 H}, \tag{20}$$

where H is the Hubble expansion rate. We have adopted the Hamilton-Jacobi formalism and a prime denotes derivatives with respect to the scalar field ϕ. The absolute values of these parameters are assumed to be less than one during inflation.

To the lowest order in these parameters, the cosmological observables are

$$P_s = \frac{1}{8\pi^2 M_p^2} \frac{H^2}{c_s \tilde{\epsilon}}, \tag{21}$$

$$P_t = \frac{2}{\pi^2} \frac{H^2}{M_p^2}, \tag{22}$$

$$n_s - 1 = 2\tilde{\eta} - 4\tilde{\epsilon} - 2\tilde{s}, \tag{23}$$

$$n_t = -2\tilde{\epsilon}, \tag{24}$$

$$r = 16 c_s \tilde{\epsilon}, \tag{25}$$

$$f_{\rm NL} = \frac{1}{3} \left(\frac{1}{c_s^2} - 1 \right), \tag{26}$$

where P_s: scalar perturbation, P_t: tensor perturbation, n_s: scalar spectral index, n_t: tensor spectral index, r: tensor to scalar ratio, $f_{\rm NL}$: non-Gaussianity parameter. Note that the right hand sides should be estimated at the moment of sound horizon crossing $kc_s = aH$ (although the tensor perturbations freeze when $k = aH$, the difference is unimportant to lowest order in the DBI parameters).

Since DBI inflation can be described as a type of brane inflation with a large Lorentz factor γ, it is clear from (17) and (26) that DBI inflation generates large non-Gaussianity. In this paper, we investigate various consistency

relations under the assumption that $|f_{\rm NL}|$ is large. For a detailed discussion on the required largeness of $|f_{\rm NL}|$, see subsection IV A.

Throughout this paper, the following relation among the observables is frequently used

$$\frac{\pi^2}{16} r^2 P_s \left(1+\frac{1}{3f_{\rm NL}}\right) = \frac{T(\phi)}{M_p^4}. \tag{27}$$

Note that ϕ in the right hand side is estimated at the moment the fluctuation is being produced. Other useful relations are

$$r = \frac{8}{M_p^2}\left(\frac{d\phi}{d\mathcal{N}}\right)^2 \tag{28}$$

where $\mathcal{N}$ is the number of e-folds, and

$$1-n_s = 4\tilde{\epsilon} + \frac{2\tilde{s}}{1-\gamma^2} - \frac{2M_p^2}{\gamma}\frac{T'H'}{TH}. \tag{29}$$

In particular, the following corollary of (28) is called the Lyth bound [13]:

$$\left(\frac{\Delta\phi}{M_p}\right)^2 \simeq \frac{r}{8}(\Delta\mathcal{N})^2. \tag{30}$$

IV. REVIEW OF CONSTRAINTS ON DBI INFLATION DRIVEN BY D3-BRANES

Constraints on gravitational waves for DBI inflation ($f_{\rm NL} \gg 1$) with a D3-brane have been derived by Lidsey and Huston (LH) [11], following the work of Baumann and McAllister (BM) [9]. We quickly review the discussion in [11] in this section. Since D3-branes are the focus of this section, (9) and (10) are simply

$$d\phi = T_3^{1/2} d\rho, \tag{31}$$
$$T = T_3 h^4. \tag{32}$$

A. Lower Bound of r

The following relation can be obtained from (29),

$$1-n_s = \frac{r}{4}\sqrt{1+3f_{\rm NL}} - \frac{2\tilde{s}}{3f_{NL}} + \frac{\dot{T}}{TH}. \tag{33}$$

Assuming the following inequality, a lower bound for the tensor to scalar ratio r can be derived,

$$\dot{T} \le 0. \tag{34}$$

From (32), this is equivalent to $\dot{h} \le 0$. In other words, this states that the brane is moving towards the tip of the throat.

The identity (33) combined with the inequality (34) gives an inequality relation,

$$\frac{r}{4}\sqrt{1+3f_{\rm NL}} - \frac{2\tilde{s}}{3f_{\rm NL}} \ge 1-n_s. \tag{35}$$

We focus on DBI inflation models generating large non-Gaussianity $|f_{\rm NL}|$ and a red spectral index $n_s < 1$. (The tilt of the spectrum is preferred to be red by the WMAP3 data. However, if there is significant negative running in the spectral index, a blue tilted spectrum is also allowed.)

When r is negligible, then the WMAP3 result $1-n_s > 0.037$ combined with (35) requires $|\tilde{s}|$ to be large ($|\tilde{s}| > 0.05|f_{\rm NL}|$) and this violates the derivation of an almost scale invariant power spectrum (23).[1] When r is non-negligible,

[1] Furthermore, since higher order terms in DBI parameters are omitted in deriving the results in Section III, a large $|\tilde{s}|$ will lead to important corrections to other cosmological observables as well. Besides, omission of higher derivative terms of ϕ in the DBI action may be inconsistent when $|\tilde{s}|$ is large.

a lower bound on r can be obtained from (35),

$$r \gtrsim \frac{4(1-n_s)}{\sqrt{1+3f_{\rm NL}}} > \frac{1-n_s}{8} \simeq 0.002. \tag{36}$$

The second inequality comes from the WMAP3 limit $|f_{\rm NL}| < 300$ [28, 29]. The far right hand side is obtained by substituting the WMAP3 best-fit value $1-n_s \simeq 0.013$.

We should remark that the second term of the left hand side of (35) was ignored in deriving the first inequality of (36). This procedure is valid under a small $\tilde{s}$ and a large $f_{\rm NL}$. For example, when $|\tilde{s}| \lesssim 0.1$, $|f_{\rm NL}| \gtrsim 20$ is sufficient.

B. Upper Bound of r

Since the 4-dimensional reduced Planck mass $M_p \equiv (8\pi G)^{-1/2}$ is given by

$$M_p^2 = \frac{2}{(2\pi)^7 g_s^2 \alpha'^4} \int d\rho \, \mathrm{Vol}(X_5) \frac{\rho^5}{h^4(\rho)}, \tag{37}$$

it is convenient to define the warped volume of extra dimensions as

$$V_6 \equiv \int d\rho \, \mathrm{Vol}(X_5) \frac{\rho^5}{h^4(\rho)}. \tag{38}$$

Here, $\mathrm{Vol}(X_5)$ is the dimensionless volume of the unit-radius 5-dimensional base space (X_5) of the throat. Generically, we expect $\mathrm{Vol}(X_5)$ to be $O(1) \times \pi^3$ (e.g. $\mathrm{Vol}(S^5) = \pi^3$ for a 5-sphere, $\mathrm{Vol}(T^{1,1}) = \frac{16}{27}\pi^3$ for a Klebanov–Strassler (KS) throat which is discussed in Appendix A).

The two inequalities used to derive the upper bound of r are the following:

$$\rho_* > \Delta\rho, \tag{39}$$

$$V_6 > \Delta V_6, \tag{40}$$

where the subscript "$*$" denotes the quantity to be estimated at the moment the CMB-scale fluctuation is produced, $\Delta\rho$ denotes the change of the D3-brane radial position when the observable scales are generated, and ΔV_6 is a fraction of the warped volume of the throat corresponding to the radial variation $\Delta\rho$. The validity of the inequalities (39) and (40) is clear for DBI inflation driven by a D-brane moving toward the tip of the throat.

Since $\Delta\rho$ corresponds to no more than $\Delta\mathcal{N} \simeq 4$ e-foldings of inflationary expansion (which is equivalent to the range $2 \leq l < 100$), $\Delta\rho$ is expected to be a narrow range in the radial dimension. Hence we adopt the following approximate expression for ΔV_6:

$$\Delta V_6 \simeq \mathrm{Vol}(X_5) \frac{\rho_*^5}{h_*^4} \Delta\rho, \tag{41}$$

where $h_* \equiv h(\phi_*)$.

From (40) and (41), we obtain

$$\frac{1}{M_p^2} = \frac{(2\pi)^7 g_s^2 \alpha'^4}{2V_6} < \frac{(2\pi)^7 g_s^2 \alpha'^4}{2\Delta V_6} \simeq \frac{(2\pi)^7 g_s^2 \alpha'^4 h_*^4}{2\mathrm{Vol}(X_5)\rho_*^5 \Delta\rho}. \tag{42}$$

This can be converted further with the use of (39),

$$\frac{1}{M_p^2} < \frac{(2\pi)^7 g_s^2 \alpha'^4 h_*^4}{2\mathrm{Vol}(X_5)(\Delta\rho)^6}. \tag{43}$$

From the Lyth bound (30), together with (27), (31), (32), and (43), an upper bound for r can be obtained

$$r < \frac{2^5 \pi^3}{(\Delta\mathcal{N})^6 \mathrm{Vol}(X_5)} P_s \left(1 + \frac{1}{3f_{\rm NL}}\right). \tag{44}$$

Taking $P_s = 2.5 \times 10^{-9}$ (WMAP3 [28] normalization), $\mathrm{Vol}(X_5) = \pi^3$, $\Delta\mathcal{N} = 1$ (the most optimistic estimate for the minimum number of e-foldings that can be probed by observation), and ignoring the $f_{\rm NL}^{-1}$ term since we have assumed

$|f_{\rm NL}|$ to be large, the upper bound becomes $r < 10^{-7}$. This obviously contradicts with the lower bound derived in the previous subsection. (Note that this upper bound is valid even if the D3-brane is moving away from the tip of the throat, as long as the inequalities (39) and (40) hold.) Therefore, DBI inflation driven by a D3-brane in relativistic motion (leading to a large $|f_{\rm NL}|$) always contradicts current observations. In other words, the above results predict low velocity of the D-brane and small $|f_{\rm NL}|$ for inflation driven by a D3-brane in a warped throat, which makes the model indistinguishable from ordinary slow-roll inflation models.

C. Note on the Difference Between BM and LH

In this section, we have briefly reviewed the constraints on gravitational waves investigated by LH. Before ending this section, we should point out some of the main differences between the approaches taken by BM [9] and LH [11].

The first is the derivation of the universal lower bound on r by LH, which is reviewed in subsection IV A. LH combined this bound with other constraints derived by BM.

Another difference is that BM considers the total variation of the D3-brane radial position *throughout inflation*, which leads to the introduction of the effective number of e-foldings

$$\mathcal{N}_{\rm eff} \equiv \int_0^{\mathcal{N}_{\rm end}} d\mathcal{N} \left(\frac{r}{r_*} \right)^{1/2}. \tag{45}$$

Since it is difficult to estimate the values of cosmological observables on scales we haven't observed, some assumptions need to be imposed in order to have a quantitative discussion on $\mathcal{N}_{\rm eff}$. In contrast, LH only make use of the variation of the D3-brane position $\Delta\rho$ and number of e-foldings $\Delta\mathcal{N}$ while the observable scales are generated, as can be seen from (39) and (40). Hence the approach taken by LH provides more conservative bounds which can be applied to general cases.

Furthermore, the constraints by LH apply to the case in which D-branes are moving relativistically, since their derivation rely on the assumption that $|f_{\rm NL}|$ is large. On the other hand, the bounds by BM also apply in the slow roll limit.

It should also be noted that the results of LH are insensitive to the details of the throat geometry and the inflaton potential. The results are directly related to cosmological observables in order to derive constraints. Meanwhile, BM consider an explicit case in which the geometry of the throat is $AdS_5 \times X_5$ and the inflaton potential is quadratic $V(\phi) = \frac{1}{2} m^2 \phi^2$. This procedure enables detailed arguments involving microscopic string theory inputs. As can be seen in [9], rather stringent bounds on the background flux can be obtained when the inflaton potential consists only of a quadratic term.

Throughout this paper we generalize the approach taken by LH and do not fix the inflaton potential to any form. (For the warp factor, a throat with $AdS_5 \times X_5$ geometry is considered as an example in Section VI.)

V. EXTENSION TO D5- AND D7-BRANES

We now consider higher dimensional D-branes and extend the bounds on r derived in the previous section for a D3-brane to the case of generic D$(3+2n)$-branes. However, derivation of upper bounds of r is complicated and subtle for D5- and D7-branes. Thus, to make arguments simpler, in this paper we consider two extreme cases in which G_{kl} or B_{kl} is dominant over the other in $\det(G_{kl} - B_{kl})$.

In this section, we derive constraints without specifying the explicit forms of the warp factor and the overall ρ-dependence of the B_2 potential. While we derive a general lower bound of r in subsection V A, the upper bound derived in subsection V B holds only in the region of the warped throat where the effect of B_2 can be ignored. There we find that the upper bound of r relaxes significantly, due to the change in the relation between the inflaton and the D-brane radial position. It is shown that there is a regime of r consistent with both the lower and upper bounds.

In the next section we specify the warp factor and the B_2 potential and provide a more complete and stringent discussion on the constraints, taking into account the tadpole condition and the known maximal value of the Euler number of a Calabi–Yau four-fold.

A. General Lower Bound of r

The derivation of the lower bound of the tensor to scalar ratio r in the previous section relies only on the inequality (34). In this section, however, for the definition of ϕ and T we now have (9) and (10) instead of (31) and (32). Thus,

in this subsection we will derive the lower bound on r by showing the inequality (34) for (9) and (10) with $n = 1, 2$. We assume that the brane is moving towards the tip of the throat, i.e. $\dot{\rho} < 0$. Thus, the inequality (34) is equivalent to

$$\frac{d}{d\rho}\left\{\int d^{2n}\xi\sqrt{h^8\det(G_{kl}-B_{kl})}\right\} \geq 0. \tag{46}$$

Let us take the angular brane coordinates to diagonalize G_{kl}. If the adequate gauge cannot be chosen throughout the $2n$-cycle, then we divide the wrapped cycle into patches on which proper coordinates can be chosen, and sum up. It is evident that G_{kl} is a Riemannian metric and, thus, the diagonal components G_{ll} are positive. Now, we rewrite G_{kl} as

$$G_{kl} = \frac{\rho^2}{h^2}\mathrm{diag}(\mathcal{G}_5, \cdots, \mathcal{G}_{4+2n}), \tag{47}$$

where $\mathcal{G}_k$ (> 0) are assumed to satisfy $\frac{d}{d\rho}(\rho^2\mathcal{G}_k) \geq 0$. (For example, this inequality is trivially satisfied if $g^{(5)}_{mn}\frac{\partial x^m}{\partial\xi^k}\frac{\partial x^n}{\partial\xi^l}$ are independent of ρ.) We note that throughout this paper, we focus on a throat with its warp factor obeying

$$\frac{dh}{d\rho} \geq 0. \tag{48}$$

For B_{kl}, we assume that it can be decomposed into the following form,

$$B_{kl} = B\mathcal{B}_{kl}, \quad \text{with} \quad \frac{dB}{d\rho} \geq 0. \tag{49}$$

Here, B is a function of ρ, while $\mathcal{B}_{kl}$ depends only on the angular brane coordinates. (For example, the fields can take the form of (47) and (49) in the region away from the tip in the KS solution, as can be seen from (A19) and (A22).)

For a D5-brane ($n = 1$), we obtain

$$h^8\det(G_{kl}-B_{kl}) = h^4\rho^4\mathcal{G}_5\mathcal{G}_6 + h^8B^2(\mathcal{B}_{56})^2. \tag{50}$$

Since each term takes the form of a product of non-decreasing functions of ρ, (46) is obvious.

For the case of a D7-brane ($n = 2$), from the discussion in Appendix D, the following can be obtained,

$$\begin{aligned} h^8\det(G_{kl}-B_{kl}) \;=\; & \rho^8\mathcal{G}_5\mathcal{G}_6\mathcal{G}_7\mathcal{G}_8 + h^4\rho^4B^2\left[\mathcal{G}_5\mathcal{G}_6(\mathcal{B}_{78})^2 + \mathcal{G}_5\mathcal{G}_7(\mathcal{B}_{68})^2 + \right.\\ & \left.\mathcal{G}_5\mathcal{G}_8(\mathcal{B}_{67})^2 + \mathcal{G}_6\mathcal{G}_7(\mathcal{B}_{58})^2 + \mathcal{G}_6\mathcal{G}_8(\mathcal{B}_{57})^2 + \mathcal{G}_7\mathcal{G}_8(\mathcal{B}_{56})^2\right]\\ & + h^8B^4(\mathcal{B}_{58}\mathcal{B}_{67} - \mathcal{B}_{57}\mathcal{B}_{68} + \mathcal{B}_{56}\mathcal{B}_{78})^2. \end{aligned} \tag{51}$$

Similarly, (46) is clear.

Hence the condition (46) is verified for both D5- and D7-branes, which leads to the lower bound (36).

Before closing this subsection, we should remark that near the tip of a deformed conifold, B_{kl} does not take the form of (49). Nevertheless, (46) and the lower bound (36) may still hold. We show this in Appendix B through the example of the KS solution.

B. Upper Bound of r in G_{kl} Dominant Region

As the D5- or D7-brane moves toward the tip of the throat, the contributions of G_{kl} and B_{kl} to $\det(G_{kl} - B_{kl})$ change. For the case of the KS solution, initially B_{kl} is dominant over G_{kl}, and then G_{kl} becomes dominant in the region closer to the tip (for a detailed discussion, see Appendix C).

As already stated in the beginning of this section, in this subsection we seek an upper bound of r without specifying the form of the warp factor h. This is possible if we can neglect B_{kl} compared with G_{kl}. This is equivalent to restricting our consideration to the region of the throat where G_{kl} is dominant over B_{kl}. The other extreme case, i.e. the B_{kl} dominant region, will be considered in the next section by using the explicit form of the warp factor and the NS-NS 2-form.

Ignoring B_{kl}, the integral term in (9) or (10) represents the wrapped volume. Introducing

$$v_{2n} \equiv \int d^{2n}\xi\sqrt{\det\left(g^{(5)}_{mn}\frac{\partial x^m}{\partial\xi^k}\frac{\partial x^n}{\partial\xi^l}\right)}, \tag{52}$$

which is the unit-radius dimensionless volume of the $2n$-cycle, (9) and (10) transform to

$$d\phi \;=\; T_{3+2n}^{1/2} v_{2n}^{1/2} \left(\frac{\rho}{h}\right)^n d\rho, \tag{53}$$

$$T(\phi) \;=\; T_{3+2n} v_{2n} h^4 \left(\frac{\rho^2}{h^2}\right)^n . \tag{54}$$

Now we combine two inequality relations (39) and (42) to obtain an upper bound on r. We transform (39) into the form

$$\left(\frac{1}{\rho_*}\right)^{\frac{2\kappa-3n+10}{n+2}} < \left(\frac{1}{\Delta\rho}\right)^{\frac{2\kappa-3n+10}{n+2}} . \tag{55}$$

Here we have introduced an arbitrary parameter κ which satisfies $\frac{2\kappa-3n+10}{n+2} > 0$ (hence κ can be taken as any nonnegative number). Later on, we will fix κ to an appropriate value in order to derive the most effective bounds. The combination of κ and n in (55) is chosen so that the final inequality expression (59) will contain an equal number of ρ_* and $1/h_*$.

Employing the inequality relation (55) on the far right hand side of (42), we obtain

$$\frac{1}{M_p^2} < \frac{(2\pi)^7\, g_s^2\, \alpha'^4}{2\,\mathrm{Vol}(X_5)} \frac{h_*^4}{\rho_*^{5-\frac{2\kappa-3n+10}{n+2}}\, \Delta\rho} \left(\frac{1}{\rho_*}\right)^{\frac{2\kappa-3n+10}{n+2}} < \frac{(2\pi)^7\, g_s^2\, \alpha'^4}{2\,\mathrm{Vol}(X_5)} \frac{h_*^4}{\rho_*^{5-\frac{2\kappa-3n+10}{n+2}}\, \Delta\rho} \left(\frac{1}{\Delta\rho}\right)^{\frac{2\kappa-3n+10}{n+2}} . \tag{56}$$

From the Lyth Bound (30) and (53),

$$\Delta\rho \simeq \frac{\Delta\mathcal{N}\, M_p\, r^{1/2}}{2^{3/2}\, T_{3+2n}^{1/2}\, v_{2n}^{1/2}} \left(\frac{h_*}{\rho_*}\right)^n . \tag{57}$$

From (27) and (54),

$$r = \frac{4\, T_{3+2n}^{1/2}\, v_{2n}^{1/2}}{\pi\, P_s^{1/2}\, (1+\frac{1}{3f_{\mathrm{NL}}})^{1/2}} \left(\frac{h_*}{M_p}\right)^2 \left(\frac{\rho_*}{h_*}\right)^n . \tag{58}$$

After cancelling out $\Delta\rho$ from the far right hand side of (56) with the use of (57), and then cancelling out M_p with the help of (58), an upper bound for r with the following form can be derived:

$$r < \frac{(2\pi)^7\, g_s^2\, \alpha'^4}{2\,\mathrm{Vol}(X_5)} \left(\frac{4\, T_{3+2n}^{1/2}\, v_{2n}^{1/2}}{\pi\, P_s^{1/2}\, (1+\frac{1}{3f_{\mathrm{NL}}})^{1/2}}\right)^{\frac{-\kappa+2n-4}{n+2}} \left(\frac{8\, T_{3+2n}\, v_{2n}}{(\Delta\mathcal{N})^2}\right)^{\frac{\kappa-n+6}{n+2}} \left(\frac{\rho_*}{h_*}\right)^{\kappa} . \tag{59}$$

It can be shown that this equation also applies to the case of $n = 0$ (when $n = \kappa = 0$, (59) turns out to be (44)). The above form of the upper bound containing equal numbers of ρ_* and $1/h_*$ is advantageous, as we will see in the subsequent discussions.

In order to obtain a bound without specifying a concrete form of the warp factor $h(\rho)$, we take κ to be 0. Again we substitute the following values, which lead to the most optimistic upper bounds (as is explained below (44)): $P_s = 2.5 \times 10^{-9}$, $\mathrm{Vol}(X_5) = \pi^3$, $\Delta\mathcal{N} = 1$, and ignoring the f_{NL} term, we obtain the following upper bounds of r

$$\text{(D5)} \qquad r < \frac{2^3\, \pi\, g_s^{2/3}\, v_2^{4/3}}{\mathrm{Vol}(X_5)\, (\Delta\mathcal{N})^{10/3}} \left\{ P_s \left(1+\frac{1}{3f_{\mathrm{NL}}}\right)\right\}^{1/3} \simeq 1.1 \times 10^{-3}\, g_s^{2/3}\, v_2^{4/3}, \tag{60}$$

$$\text{(D7)} \qquad r < \frac{4\, g_s\, v_4}{\mathrm{Vol}(X_5)\, (\Delta\mathcal{N})^2} \simeq 0.13\, g_s\, v_4. \tag{61}$$

We expect v_{2n} to obey $v_2 \sim 4\pi$ (which is the value for a 2-sphere) and $v_4 \sim \frac{8}{3}\pi^2$ (the value for a 4-sphere). Hence, as long as the string coupling constant g_s is larger than $\mathcal{O}(10^{-2})$ for a D5- and $\mathcal{O}(10^{-4})$ for a D7-brane, the upper bound for r is compatible with the lower bound.

One may expect to obtain more stringent bounds by assigning some number other than 0 to κ in (59). However, in order to do so, the explicit form of the warp factor is needed. In view of this, in the following section, we focus on a warped throat with an AdS geometry and derive more severe constraints.

VI. MORE STRINGENT BOUNDS

In the previous section we found that the upper bounds of r for D5- and D7-branes are significantly weaker than that for a D3-brane and are compatible with the lower bound. However, in this section we will find that a large number of background charge is needed. In many cases this requires too large an Euler number of a Calabi–Yau four-fold, well beyond the known maximal value.

To make our arguments concrete, in this section we assume that the throat is approximately $\mathrm{AdS}_5 \times X_5$ away from the tip. Though the throat geometry deviates from the AdS geometry as one approaches the tip for the case of a deformed conifold, we focus on the AdS region of the warped throat and estimate various constraints. The validity of this procedure is shown in Appendix B, where it is shown through the example of a KS throat that the observed CMB scale is generated away from the tip in DBI inflation.

The warp factor in the AdS region is $h(\rho) = \rho/R$ with the AdS radius

$$\frac{\rho}{h(\rho)} = R = \left(\frac{2^2 \pi^4 g_s \alpha'^2 N}{\mathrm{Vol}(X_5)} \right)^{1/4}, \tag{62}$$

where N is the background number of charges [30].

The warp factor at the tip of the throat is characterized by the integers M and K associated with R-R and NS-NS fluxes respectively, as [31]

$$h(0) \sim \exp\left(-\frac{2\pi K}{3 g_s M} \right). \tag{63}$$

As we briefly mentioned in Section II, in many cases, the NS-NS 2-form potential B_2 depends on the radial coordinate logarithmically in the AdS region of the throat and have legs along the angular directions of the internal space. Therefore let us consider the case of B_2 taking the following form [17, 18]:

$$B_{kl} = B \mathcal{B}_{kl}, \quad B = g_s M \alpha' \ln\left(\frac{\rho}{\rho_b} \right). \tag{64}$$

where M is an integer associated with R-R 3-form F_3, and $\mathcal{B}_{kl}$ is independent of ρ as in (49). For example, as can be seen in (A22), the KS solution in the large τ region indeed has B_{kl} of this form. We assume ρ_b to be roughly the place of the boundary between the AdS region and the region near the tip where the warp factor is nearly constant. Then ρ_b takes the following form:

$$\rho_b \sim (g_s M \alpha')^{1/2} \exp\left(-\frac{2\pi K}{3 g_s M} \right), \tag{65}$$

where K is an integer associated with the NS-NS 3-form H_3. (For the value of ρ_b in the KS solution, see (A23).)

The product of M and K produces the net background charge N, and from the tadpole condition it is related to the topology of a Calabi–Yau four-fold:

$$KM = N = \frac{\chi}{24} \leq 75852, \tag{66}$$

where χ is the Euler characteristic of a CY four-fold and the inequality on the right hand side comes from the known maximal value $\chi = 1820448$ [14].

As already stated in the previous section, derivation of the upper bounds of r for D5- and D7- branes is rather complex because of the presence of the determinant of $(G_{kl} - B_{kl})$ in the action. Therefore we consider two extreme cases in which G_{kl} or B_{kl} is dominant over the other. In the following subsections, we consider each case separately.

Moreover, in Appendix C, the number of e-foldings generated in the G_{kl} dominant region is roughly estimated and we discuss the place where the CMB scale is produced.

A. Lower Bound of N in G_{kl} Dominant Region

In this subsection we consider the case in which G_{kl} is dominant over B_{kl}. As discussed in Appendix C, this corresponds to the region relatively closer to the tip.

We have already considered this region in subsection V B and obtained the upper bound of r (59). With the use of (62), we now reinterpret (59) as a lower bound of the background charge N:

$$N > \frac{\mathrm{Vol}(X_5)(\Delta\mathcal{N})^{\frac{8}{2+n}}}{2^{\frac{2(1-n)}{2+n}}\pi^{\frac{6}{2+n}}g_s^{\frac{n}{2+n}}v_{2n}^{\frac{2}{2+n}}P_s^{\frac{2}{2+n}}(1+\frac{1}{3f_{\mathrm{NL}}})^{\frac{2}{2+n}}}\left[\frac{\mathrm{Vol}(X_5)(\Delta\mathcal{N})^{\frac{2(6-n)}{2+n}}r}{2^{\frac{10-n}{2+n}}\pi^{\frac{3(2-n)}{2+n}}g_s^{\frac{2n}{2+n}}v_{2n}^{\frac{4}{2+n}}P_s^{\frac{2-n}{2+n}}(1+\frac{1}{3f_{\mathrm{NL}}})^{\frac{2-n}{2+n}}}\right]^{4/\kappa}, \tag{67}$$

where κ is now an arbitrary positive number.

Again we substitute $P_s = 2.5\times10^{-9}$, $\mathrm{Vol}(X_5)=\pi^3$, $v_2 = 4\pi$, $v_4 = \frac{8}{3}\pi^2$, $\frac{1}{3f_{\mathrm{NL}}} = 0$. Furthermore, the minimum number of e-foldings $\Delta\mathcal{N} = 1$ and the lower limit value for the tensor-to-scalar ratio during relativistic inflation $r = 0.002$ from (36) is substituted in order to obtain a conservative bound for N. For g_s, we take the value 0.1.

- D3-brane ($n = 0$)
  In Sec. IV, it was already shown that DBI inflation driven by a D3-brane is inconsistent with cosmological observations if $|f_{\mathrm{NL}}|$ is large. Nonetheless, as a consistency check, let us show it again. The bound (67) turns out to be $N > 2.0\times10^8\times(3.9\times10^{17})^{1/\kappa}$. Taking the limit $\kappa\to0+$, this lower bound for N diverges, which implies that an infinitely large N is required for a D3-brane to cause DBI inflation, i.e. DBI inflation is incompatible with WMAP3 data.[2] Hence we have obtained the same result, seen from a different perspective.

- D5-brane ($n = 1$)
  The bound (67) in this case is $N > 6.8\times10^5\times(7.0\times10^{-3})^{1/\kappa}$. Since the number in parentheses is less than one, let us take the limit $\kappa\to\infty$. The lower bound for N turns out to be 6.8×10^5. This exceeds the known maximal value of N (66) by an order.

- D7-brane ($n = 2$)
  In this case we obtain $N > 9.7\times10^4\times(1.2\times10^{-9})^{1/\kappa}$ from (67). The lower bound for N becomes 9.7×10^4 in the limit $\kappa\to\infty$. This still exceeds the maximal known value of N. However, adopting different values to the parameters may relax the lower bound. If the string coupling is larger than about 0.2 (while keeping the other parameters fixed to the values discussed above), then the lower bound for N becomes compatible with (66). The same could be done by considering a throat with $\mathrm{Vol}(X_5)\lesssim 20$.

Note that the upper bound for r (59) and the lower bound for N (67) relax with a larger string coupling. The results above imply the difficulty of maintaining perturbative control $g_s < 1$ and satisfying the upper bound for N (66) at the same time.

B. Upper Bound of r in B_{kl} Dominant Region

We now consider the opposite extreme case, where B_{kl} dominates over G_{kl}. The results we obtain in this subsection is expected to be relevant if perturbations of the CMB scale are generated in the large ρ region (see Appendix C.).

In this subsection we derive an upper bound of r. In this sense the analysis in this subsection is a counterpart to that of subsection V B, where we have derived an upper bound of r in the G_{kl} dominant region. However, the difference is that the analysis in this subsection requires (reasonable but explicit) assumptions about the properties of the warp factor $h(\rho)$ and the NS-NS flux $B(\rho)$ which were reviewed in the beginning of this section, while the analyses in subsection V B were independent of those properties.

In the next subsection we shall reinterpret the result of this subsection as an lower bound of the background charge N.

Now let us start the analysis by introducing

$$b_{2n} \equiv \int d^{2n}\xi\sqrt{\det\mathcal{B}_{kl}}. \tag{68}$$

---

[2] It was already mentioned in [6] that the original DBI scenario with D3-branes required a large number of background charge $N\gtrsim10^{10}$. This result was derived under the assumption of the inflaton potential consisting only of a quadratic term. We have shown here without specifying the form of the potential that a more stringent condition $N>\infty$ can be obtained by combining the results of Section IV.

Based on examples of specific cycles in the KS solution, we expect that $b_2 \sim v_2 \sim 4\pi$ and that $b_4 \sim v_4 \sim \frac{8}{3}\pi^2$. Then (9) and (10) now take the form

$$
\begin{aligned}
d\phi &= T_{3+2n}^{1/2} b_{2n}^{1/2} \left\{ g_s M \alpha' \ln\left(\frac{\rho}{\rho_b}\right) \right\}^{n/2} d\rho, && (69)\\
T(\phi) &= T_{3+2n} b_{2n} h^4 \left\{ g_s M \alpha' \ln\left(\frac{\rho}{\rho_b}\right) \right\}^{n}. && (70)
\end{aligned}
$$

From the Lyth Bound (30) and (69),

$$
\Delta\rho \simeq \frac{\Delta\mathcal{N}}{2^{3/2} T_{3+2n}^{1/2} (g_s M \alpha')^{n/2} b_{2n}^{1/2}} \frac{M_p r^{1/2}}{\left\{ \ln\left(\frac{\rho_*}{\rho_b}\right) \right\}^{n/2}}. \tag{71}
$$

Combining (27) with (70),

$$
r = \frac{4\, T_{3+2n}^{1/2}\, b_{2n}^{1/2}\, (g_s M \alpha')^{n/2}}{\pi\, P_s^{1/2} \left(1 + \frac{1}{3 f_{\rm NL}}\right)^{1/2}} \left(\frac{h_*}{M_p}\right)^2 \left\{ \ln\left(\frac{\rho_*}{\rho_b}\right) \right\}^{n/2}. \tag{72}
$$

From (39) and (42), and again introducing an arbitrary parameter κ as in subsection V B,

$$
\frac{1}{M_p^2} < \frac{(2\pi)^7\, g_s^2\, \alpha'^4}{2\, \mathrm{Vol}(X_5)} \frac{h_*^4\, \rho_*^\kappa}{\Delta\rho} \left(\frac{1}{\rho_*}\right)^{\kappa+5} < \frac{(2\pi)^7\, g_s^2\, \alpha'^4}{2\, \mathrm{Vol}(X_5)} \frac{h_*^4\, \rho_*^\kappa}{\Delta\rho} \left(\frac{1}{\Delta\rho}\right)^{\kappa+5}. \tag{73}
$$

Now κ has to satisfy $\kappa > -5$. $\Delta\rho$ can be cancelled out from the far right hand side of (73) with the use of (71). Then, after cancelling out M_p with the help of (72), one can deduce an upper bound for r,

$$
r < \frac{(2\pi)^7\, g_s^2\, \alpha'^4}{2\, \mathrm{Vol}(X_5)} \left(\frac{\pi^{1/2} \{P_s(1 + \frac{1}{3 f_{\rm NL}})\}^{1/4}}{2 T_{3+2n}^{1/4} b_{2n}^{1/4} (g_s M \alpha')^{n/4}} \right)^{\kappa+4} \left(\frac{\Delta\mathcal{N}}{2^{3/2} T_{3+2n}^{1/2} (g_s M \alpha')^{n/2} b_{2n}^{1/2}} \right)^{-\kappa-6} \left(\frac{\rho_*}{h_*}\right)^\kappa \left\{ \ln\left(\frac{\rho_*}{\rho_b}\right) \right\}^{\frac{n}{4}(\kappa+8)}. \tag{74}
$$

Since $\frac{\rho}{h} = R$, $\rho \sim R$ is roughly the place where the throat connects to the bulk (to be precise, one should be aware that the geometry deviates from the AdS geometry in the UV region, due to the connection of the throat to the bulk). Since we consider the case of inflation occurring within a single throat, $\rho_* < R$. Together with (62), (74) can be rewritten in the following form,

$$
r < \frac{(2\pi)^7\, g_s^2\, \alpha'^4}{2\, \mathrm{Vol}(X_5)} \left(\frac{\pi^{1/2} \{P_s(1 + \frac{1}{3 f_{\rm NL}})\}^{1/4}}{2 T_{3+2n}^{1/4} (g_s M \alpha')^{n/4} b_{2n}^{1/4}} \right)^{\kappa+4} \left(\frac{\Delta\mathcal{N}}{2^{3/2} T_{3+2n}^{1/2} (g_s M \alpha')^{n/2} b_{2n}^{1/2}} \right)^{-\kappa-6} R^\kappa \left\{ \ln\left(\frac{R}{\rho_b}\right) \right\}^{\frac{n}{4}(\kappa+8)}. \tag{75}
$$

From (62) and (65)

$$
\frac{R}{\rho_b} \sim \frac{2^{1/2}\, \pi}{\mathrm{Vol}(X_5)^{1/4}} \left(\frac{K}{g_s\, M}\right)^{1/4} \exp\left(\frac{2\pi K}{3 g_s M}\right). \tag{76}
$$

Since the exponential factor in the right hand side is approximately the inverse of the warping at the tip of the throat (63), $\frac{2\pi K}{3 g_s M}$ is expected to be larger than 1. Therefore,

$$
\ln \frac{R}{\rho_b} \sim \frac{2\pi K}{3 g_s M}. \tag{77}
$$

Substituting (77) to (75) and taking κ to 0 yields the following upper limit for r:

$$
\begin{aligned}
&(\mathrm{D5}) \qquad r < \frac{2^3\, \pi\, b_2^2\, K^2\, P_s(1 + \frac{1}{3 f_{\rm NL}})}{3^2\, \mathrm{Vol}(X_5)\, (\Delta\mathcal{N})^6} \simeq 3.5 \times 10^{-8} K^2, && (78)\\
&(\mathrm{D7}) \qquad r < \frac{2\, b_4^2\, K^4\, P_s(1 + \frac{1}{3 f_{\rm NL}})}{3^4\, \pi\, \mathrm{Vol}(X_5)\, (\Delta\mathcal{N})^6} \simeq 4.4 \times 10^{-10} K^4. && (79)
\end{aligned}
$$

Here, $P_s = 2.5 \times 10^{-9}$, $\mathrm{Vol}(X_5) = \pi^3$, $b_2 = 4\pi$, $b_4 = \frac{8}{3}\pi^2$, $\frac{1}{3 f_{\rm NL}} = 0$, and $\Delta\mathcal{N} = 1$ have been substituted for the estimate of the far right hand sides.

For these upper bounds to be compatible with the lower bound (36), K needs to be larger than about 240 for a D5- and 46 for a D7-brane, which can readily be achieved.

C. Lower Bound of N in B_{kl} Dominant Region

The upper bound of r (75) together with (77) can be transformed into a lower bound for N with $\kappa > 0$:

$$N > \frac{2^{n-1}3^n \mathrm{Vol}(X_5)(\Delta\mathcal{N})^4}{\pi^{3-n} b_{2n} K^n P_s(1+\frac{1}{3f_{\mathrm{NL}}})} \left(\frac{2^{2n-5}\, 3^{2n}\, \pi^{2n-3}\, \mathrm{Vol}(X_5)\, (\Delta\mathcal{N})^6\, r}{b_{2n}^2\, P_s(1+\frac{1}{3f_{\mathrm{NL}}}) K^{2n}} \right)^{\frac{4}{\kappa}} \tag{80}$$

Substituting the same values as above and $r = 0.002$, the following results can be obtained:

- D5-brane ($n = 1$)
  $NK > 3.0 \times 10^8 \times (240/K)^{8/\kappa}$. When $K \gtrsim 240$, taking $\kappa \to \infty$ yields a lower bound for the combination of the charge numbers, $NK > 3.0 \times 10^8$. For example, when the background charge takes the maximum value $N = 75852$, the lower bound requires $K \gtrsim 4000$, which in this case is equivalent to $M \lesssim 20$. The smallness of M may invalidate the supergravity approximation in the warped throat.

- D7-brane ($n = 2$)
  $NK^2 > 2.7 \times 10^9 \times (46/K)^{16/\kappa}$. If $K \gtrsim 46$, then taking $\kappa \to \infty$ yields $NK^2 > 2.7 \times 10^9$. When $N = 75852$, the constraint turns out to be $K \gtrsim 190$ ($M \lesssim 4000$).

In terms of the background charge, a D7-brane doesn't seem to have problems exciting DBI inflation. We also remark that the upper bound for r (78) and (79), and the lower bound for N (80) do not depend on the string coupling directly, in contrast to the bounds derived in the G_{kl} dominant region.

VII. CONCLUSION

We have presented constraints on DBI inflation with large non-Gaussianity, caused by a D-brane in relativistic motion towards the tip of the throat. We focused on D5- and D7-branes which wrap cycles of the warped throat. As expected, the upper bound on the gravitational wave spectrum for D3-branes is relaxed for wrapped D5- and D7-branes, due to the difference in the normalization factor of the inflaton. However, for when the known maximal value for the Euler number of the Calabi–Yau four-fold is considered, we showed the difficulty in obtaining a sufficient number of background charge for producing the cosmological observables consistent with the WMAP3 data. The results of this paper are insensitive to the details of the inflaton potential, and could be applied to generic warped throats.

Our estimation imposes severe constraints on a D5-brane turning on DBI inflation. However, for the case of a D7-brane producing the CMB scale in the B_{kl} dominant region of the throat, the constraint is loosened. D7-brane DBI inflation in the G_{kl} dominant region may also be compatible with the Euler number bound, provided the string coupling and the parameters of the throat are tuned to some appropriate range. These cases may be the area to seek for brane inflation models generating observable signals in the sky.

Our aim in this paper was to estimate whether DBI inflation can reproduce observable signatures compatible with WMAP3 data in the currently known construction of string theory. The results in this paper indicate the difficulty of obtaining workable models due to the Euler number bound. However, let us mention some possible scenarios in which the bound may be relaxed. Though it is presently unknown whether those possibilities can be implemented in string theory, they are worth pointing out. One way of alleviating the Euler number bound is to consider a throat with a base space that is orbifolded [6, 34]. Orbifolding implies a smaller volume of the base space $\mathrm{Vol}(X_5)$, leading to a relaxation of the lower bound on N ((67) and (80)). If the orbifolded base space decreases the volume of the wrapped cycles (which works in the direction of tightening the bound), one can also consider D-branes winding around the throat more than once to cancel the decrease of v_{2n}. However, it is obvious from the results in this paper that a substantial number of orbifolding (and winding number) is needed in any case. At present it is far from clear whether this could be justified in a consistent compactification. Other ways to relax the Euler number bound are to add negative charges by considering orientifolds or gluing on a whole throat of negative D3 charge [35] (though this would lead to an increase of the compactified volume).[3] Finally, if the Euler number of Calabi–Yau four-folds exceeding the known maximal value is to be found, then of course the constraints derived in this paper will be relaxed.

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[3] We thank Melanie Becker and Sarah Shandera for pointing these out to us.

An alternative approach to bypassing the background flux bounds is to combine DBI inflation with other scenarios such as modular inflation models. Then DBI inflation could generate large non-Gaussianity while some other sector is producing enough e-foldings.

We should also mention backreaction of the mobile D-branes on the background geometry. We treated wrapped D-branes as a probe, but since D5- and D7-branes have a more significant backreaction than D3-branes (especially when the winding number is large), this potential problem deserves careful consideration. An order of magnitude estimate of the backreaction by multi-winding D5-branes around orbifolded base space is given in [34].

We did not fix the form of the inflaton potential throughout this paper. However, as we mentioned in subsection IV C, it should be noted that under some explicit form of potentials, even more severe constraints on the background charge could be obtained (see for e.g. [9, 34] where a tight upper bound on N is derived under the assumption of a quadratic inflaton potential). The constraints we investigated are conservative bounds, which can be applied to general situations.

Acknowledgments

T.K. and S.K. would like to thank Katsuhiko Sato for his encouragement and support. S.K.'s work was in part supported by JSPS through a Grant-in-Aid for JSPS Fellows. S.M. was supported by MEXT through a Grant-in-Aid for Young Scientists (B) No. 17740134, and by JSPS through a Grant-in-Aid for Creative Scientific Research No. 19GS0219 and through a Grant-in-Aid for Scientific Research (B) No. 19340054.

APPENDIX A: KLEBANOV–STRASSLER SOLUTION

The KS geometry [15, 16] is

$$ds^2 = \tilde{h}^{-1/2}(\tau)\eta_{\mu\nu}dx^\mu dx^\nu + \tilde{h}^{1/2}(\tau)ds_6^2 \tag{A1}$$

where x^μ $(\mu = 0, \cdots 3)$ are 4-dimensional external coordinates and ds_6^2 is the metric of the deformed conifold [32]

$$ds_6^2 = \frac{\epsilon^{4/3}}{2}K(\tau)\left[\frac{1}{3K(\tau)^3}\Big(d\tau^2 + (g^5)^2\Big) + \cosh^2\Big(\frac{\tau}{2}\Big)\Big((g^3)^2 + (g^4)^2\Big) + \sinh^2\Big(\frac{\tau}{2}\Big)\Big((g^1)^2 + (g^2)^2\Big)\right] \tag{A2}$$

where

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh\tau} \tag{A3}$$

and g^i $(i = 1, \cdots 5)$ are orthonormal basis [33] defined by

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}}, \quad g^3 = \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}}, \quad g^5 = e^5 \tag{A4}$$

with

$$\begin{aligned} e^1 = -\sin\theta_1\, d\phi_1, \qquad e^2 = d\theta_1, \qquad e^3 = \cos\psi \sin\theta_2\, d\phi_2 - \sin\psi\, d\theta_2, \\ e^4 = \sin\psi\sin\theta_2\, d\phi_2 + \cos\psi\, d\theta_2, \qquad e^5 = d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2. \end{aligned} \tag{A5}$$

Away from the tip, the geometry is approximately $\mathrm{AdS}_5 \times S^5$.

The base of the cone has the topology of $S^2 \times S^3$. As one approaches the tip, the radius of S^2 shrinks to zero, while the radius of S^3 remains finite. Hence, the geometry is roughly $S^3 \times R^3$ at the tip of the throat. The S^3 subspace is a 3-cycle, which is referred to as the A-cycle. Another dual 3-cycle which is the S^2 times a circle extending along the radial direction is called the B-cycle. The R-R 3-form flux F_3 and NS-NS 3-form flux H_3 is supported on these cycles, whose quantization conditions are

$$\frac{1}{2\pi\alpha'}\int_A F_3 = 2\pi M, \quad \frac{1}{2\pi\alpha'}\int_B H_3 = -2\pi K, \tag{A6}$$

where M and K are integers, and

$$g_s^2 F_3^2 = H_3^2. \tag{A7}$$

The tadpole condition requires

$$KM = \frac{\chi}{24} \tag{A8}$$

where χ is the Euler number of a Calabi–Yau four-fold.

F_3 and B_2 have the Z_2 symmetric $((\theta_1,\ \phi_1) \leftrightarrow (\theta_2,\ \phi_2))$ ansatz:

$$F_3 = \frac{M\alpha'}{2}\big[g^5 \wedge g^3 \wedge g^4 + d\{F(\tau)(g^1 \wedge g^3 + g^2 \wedge g^4)\}\big], \tag{A9}$$

$$B_2 = \frac{g_s M\alpha'}{2}\big[f(\tau)g^1 \wedge g^2 + k(\tau)g^3 \wedge g^4\big]. \tag{A10}$$

Combining with (A7), the dilaton Φ and the R-R scalar C_0 can consistently be set to zero. The BPS saturated solution found by Klebanov and Strassler is

$$F(\tau) = \frac{\sinh\tau - \tau}{2\sinh\tau}, \quad f(\tau) = \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1), \quad k(\tau) = \frac{\tau\coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1), \tag{A11}$$

and

$$\tilde{h}(\tau) = 2^{2/3}(g_s M\alpha')^2\epsilon^{-8/3}I(\tau), \tag{A12}$$

where

$$I(\tau) = \int_\tau^\infty dx \frac{x\coth x - 1}{\sinh^2 x}(\sinh(2x) - 2x)^{1/3}. \tag{A13}$$

For this solution,

$$C_4 = g_s^{-1}\tilde{h}^{-1}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{A14}$$

in a particular gauge. For large $g_s M$ the curvature is small everywhere and we can trust the supergravity description.

$I(\tau)$ reaches a finite value as τ approaches zero,

$$I(0) = a_0 \ \ \text{with} \ \ a_0 \simeq 0.71805. \tag{A15}$$

Since the warp factor at the tip of the throat can be characterized by the flux integer numbers M and K introduced in (A6),

$$\tilde{h}^{-1/4}(0) \simeq \exp\left(-\frac{2\pi K}{3g_s M}\right), \tag{A16}$$

the deformation parameter ϵ in (A12) is

$$\epsilon \simeq 2^{1/4}a_0^{3/8}(g_s M\alpha')^{3/4}\exp\left(-\frac{\pi K}{g_s M}\right). \tag{A17}$$

The correspondence between the KS metric (A1) and the generic metric (1) we use in this paper is clear by the change of variables

$$d\rho = \frac{\epsilon^{2/3}}{\sqrt{6}K(\tau)}d\tau, \tag{A18}$$

and

$$h(\rho) = \tilde{h}(\tau)^{-1/4}. \tag{A19}$$

In the large τ region, the metric of the deformed metric (A2), the relation between ρ and τ (A18), and the NS-NS 2-form potential B_2 (A10) is

$$ds_6^2 \simeq d\rho^2 + \rho^2\left\{\frac{1}{6}\big((g^1)^2 + (g^2)^2 + (g^3)^2 + (g^4)^2\big) + \frac{1}{9}(g^5)^2\right\}, \tag{A20}$$

$$\rho \simeq \frac{3^{1/2}}{2^{5/6}}\epsilon^{2/3}e^{\tau/3}, \tag{A21}$$

$$B_2 \simeq \frac{3}{4}g_sM\alpha' \ln\left(\frac{\rho}{\rho_b}\right)[g^1 \wedge g^2 - g^3 \wedge g^4], \tag{A22}$$

where

$$\rho_b \equiv \frac{3^{1/2}e^{1/3}}{2^{5/6}}\epsilon^{2/3} \simeq \frac{3^{1/2}e^{1/3}a_0^{1/4}}{2^{2/3}}(g_sM\alpha')^{1/2}\exp\left(-\frac{2\pi K}{3g_sM}\right). \tag{A23}$$

The far right hand side of (A23) is from (A17).

In the small τ region ($\tau \lesssim 1$),

$$ds_6^2 \simeq d\rho^2 + \rho^2\left\{\frac{1}{2}(g^1)^2 + \frac{1}{2}(g^2)^2\right\} + \rho_0^2\left\{(g^3)^2 + (g^4)^2 + \frac{1}{2}(g^5)^2\right\} \tag{A24}$$

where

$$\rho_0 \equiv \frac{\epsilon^{2/3}}{2^{1/3}3^{1/6}}, \tag{A25}$$

$$\rho = \frac{\epsilon^{2/3}\tau}{2^{5/6}3^{1/6}}, \tag{A26}$$

$$B_2 \simeq \frac{g_sM\alpha'}{2}\left[\frac{\tau^3}{12}(g^1 \wedge g^2) + \frac{\tau}{3}(g^3 \wedge g^4)\right]. \tag{A27}$$

APPENDIX B: DBI INFLATION NEAR THE TIP OF THE THROAT

In this appendix, we discuss DBI inflation in the non-AdS region, nearby the tip of the throat. We take the KS throat as an example and show that the CMB scale is produced away from this region.

In a KS throat, the conifold singularity is smoothed out by turning on background flux. While the S^2 of the base space disappears as one approaches the tip of the deformed conifold, the S^3 remains finite and the warp factor approaches a constant value (A16).

In the region $\tau \lesssim 1$ (i.e. $\rho \lesssim 0.8\rho_0$, where ρ_0 is defined in (A25)), τ and ρ become proportional to each other, as can be seen from (A26). The metric and the NS-NS 2-form potential B_2 in the region is given by (A24) and (A27).

In the g^1 and g^2 directions, the behavior of the metric and B_2 in this region are respectively

$$\frac{\rho^2}{h^2} \propto \tau^2, \quad f(\tau) \propto \tau^3. \tag{B1}$$

In the g^3 and g^4 directions,

$$\frac{\rho_0^2}{h^2} \propto \tau^0, \quad k(\tau) \propto \tau. \tag{B2}$$

Furthermore, B_2 does not have a leg in the g^5 direction.

Hence it is expected that the B_2 term is negligible compared to the term coming from the metric in the DBI action (8). Therefore we ignore B_2 in estimating possible e-fold numbers near the tip.

Ignoring B_2, the applicability of the lower bound for the tensor to scalar ratio (36) to the region near the tip with the metric (A24) can be verified in a similar way to the discussion in subsection V A. Taking the angular brane coordinates to diagonalize G_{kl}, it is clear that $\partial(h^2G_{kk})/\partial\rho \geq 0$. Hence (46) is verified, leading to the lower bound (36).

Also, the equations introduced in subsection V B to estimate the upper bound for r in the G_{kl} dominant region can be used here likewise. However, it should be noted that the constants $\mathrm{Vol}(X_5)$ and v_{2n} now become dependent

on ρ, due to the S^2's constant radius. The effective unit-radius dimensionless volume of the wrapped $2n$-cycle is now defined as

$$v_{2n}(\rho) \equiv \left(\frac{h}{\rho}\right)^{2n} \int d^{2n}\xi \sqrt{\det(G_{kl})}. \tag{B3}$$

Now let us give a rough estimate of the number of e-foldings which can be generated in the region near the tip. We will see that the e-fold number is suppressed to a negligible amount by the warping of the throat.

A typical element of G_{kl} takes the form of a quadratic equation of ρ or ρ_0, divided by h^2. Therefore, $v_{2n}(\rho)$ can practically be expressed as a product of ρ^{-2n} and a $2n$-order polynomial of ρ or ρ_0. Therefore, in the "tip" region, i.e. $\rho \lesssim 0.8\rho_0$, the effective unit-radius dimensionless volume is

$$v_{2n}(\rho) \sim \frac{\mathcal{O}\big(\rho_0^{2n}\big)}{\rho^{2n}}. \tag{B4}$$

Combining the Lyth Bound (30), (53), and (B4), we obtain

$$d\mathcal{N} \sim \frac{T_{3+2n}^{1/2}\,\mathcal{O}\big(\rho_0^n\big)}{M_p\, r^{1/2}\, h^n} d\rho. \tag{B5}$$

The number of e-foldings can be evaluated by integrating the right hand side. We estimate the maximal number of e-foldings by fixing the tensor to scalar ratio to the smallest $r = r_{\rm min}$ in the region of integration. Similarly, we fix the warp factor to the value at the tip of the throat $h = h_{\rm tip}$. Integrating between 0 and ρ_0, (it should be noted that inflation actually ends before reaching $\rho = 0$ in most models)

$$\mathcal{N}_{\rm tip} \lesssim \frac{T_{3+2n}^{\frac{1}{2}}\,\mathcal{O}\big((\rho_0)^{n+1}\big)}{M_p\, r_{\rm min}^{\frac{1}{2}}\, h_{\rm tip}^n} \sim \frac{g_s^{\frac{n}{2}}\, M^{\frac{n+1}{2}}\, h_{\rm tip}}{\alpha'^{\frac{1}{2}}\, M_p\, r_{\rm min}^{\frac{1}{2}}}. \tag{B6}$$

In obtaining the far right hand side, we have used

$$\rho_0 = \frac{a_0^{1/4}}{2^{1/6}\, 3^{1/6}} (g_s M \alpha')^{1/2} h_{\rm tip} \tag{B7}$$

which is a combination of (A16), (A17), (A19), and (A25). Hence we have shown that the number of e-foldings that can be generated in the "tip" region is suppressed by the warp factor.

Now we confirm the above result with an explicit example. Let us consider a D5-brane which wraps a 2-cycle specified by

$$\psi = 0,\ \ \theta_1 = \theta_2,\ \ \phi_1 = -\phi_2. \tag{B8}$$

Then the following can be derived

$$\int d^2\xi \sqrt{\det(G_{lk} - B_{lk})} = \frac{2^3\, \pi\, \rho_0^2}{h^2} \left\{ 1 + \left(\frac{h^2\, g_s M \alpha' \rho}{2^{2/3}\, 3\, \rho_0^3} \right)^2 \right\}^{1/2}. \tag{B9}$$

The first term in parentheses of the right hand side originates from G_{kl}, and the second term from B_{kl}. Approximating the warp factor by the value at the tip, then from (B7) the ratio between them is

$$\frac{B\,\text{term}}{G\,\text{term}} \simeq 0.2 \times \frac{\rho^2}{\rho_0^2}. \tag{B10}$$

Hence it is clear that B_2 can be ignored in the "tip" region (i.e. $\rho \lesssim 0.8\rho_0$).

Ignoring the B_2 term, the effective unit-radius dimensionless volume is

$$v_2(\rho) = \big(\frac{\rho_0}{\rho}\big)^2 8\pi. \tag{B11}$$

This simple form originates from the fact that the 2-cycle specified by (B8) wraps only the non-vanishing S^3.

The integration which led (B5) to (B6) can be explicitly carried out for the simple form of v_2 (B11). Keeping also the numerical factors, we obtain

$$\mathcal{N}_{\rm tip} \leq \frac{0.007}{r_{\rm min}^{1/2}} \times \frac{g_s^{1/2}\, M}{M_p\, \alpha'^{1/2}} \times h_{\rm tip} \tag{B12}$$

Assuming that the CMB scale is generated in the "tip" region, we substitute $r_{\rm min} = 0.002$ which is the lower bound of r over the observable scales for DBI inflation (36). Then the right hand side of (B12) is expected to be dominated by the warping at the tip of the throat, suppressing $\mathcal{N}_{\rm tip}$ to be much smaller than the minimum number of e-foldings produced while the observable scales are generated, i.e. $\Delta\mathcal{N} \simeq 1$.

Since the CMB scale was generated 30 to 60 e-foldings before the end of inflation, the above estimates indicate that the CMB scale cannot be produced in the non-AdS region nearby the tip of the KS throat.

Throughout this paper we consider warped throats. However, we should note that in the case of a barely warped throat (e.g. $h_{\rm tip} \sim \mathcal{O}(1)$), there will be a large non-AdS region and it is possible that a sufficiently large number of e-foldings will be generated in the region. (In that case, the volume of the internal space must be large for explaining the hierarchy of the universe.)

APPENDIX C: NUMBER OF e-FOLDINGS IN THE G_{kl} DOMINANT REGION

In Sections V and VI we discussed constraints on r and N for the case of the observed CMB scale being produced in the G_{kl} or B_{kl} dominant region. The analyses indicate that DBI inflation requires a large Euler number of the Calabi–Yau four-fold, which exceeds the maximal value (66) under typical values for various parameters. However, for DBI inflation with a D7-brane in the B_{kl} dominant region, the constraint is relaxed.

In this appendix, we roughly estimate the number of e-foldings than can be produced in the G_{kl} dominant region and consider the place where the CMB scale is produced.

The typical ratio between the components of G_{kl} and B_{kl} is

$$\frac{\text{components of } G_{kl}}{\text{components of } B_{kl}} = \frac{\rho^2}{h^2} \times \left\{ g_s M \alpha' \ln\left(\frac{\rho}{\rho_b}\right) \right\}^{-1} = \frac{R^2}{g_s M \alpha' \ln(\frac{\rho}{\rho_b})}. \tag{C1}$$

Therefore, the G_{kl} dominant region is roughly

$$\rho < \rho_b \exp\left(\frac{R^2}{g_s M \alpha'}\right) \equiv \rho_{G/B}. \tag{C2}$$

From the Lyth Bound (30), (62), and (53),

$$d\mathcal{N} \simeq \frac{v_{2n}^{1/2}\, N^{n/4}}{2^{n/2}\, \pi^{3/2}\, g_s^{(2-n)/4}\, \alpha'\, M_p\, \mathrm{Vol}(X_5)^{n/4}\, r^{1/2}}\, d\rho. \tag{C3}$$

The number of e-foldings $\mathcal{N}_G$ generated in the G_{kl} dominant AdS region can be obtained by integrating (C3) between $\rho_{G/B}$ and the IR boundary of the AdS region, which we approximate to 0. Note that r depends on ρ. Combining (C1) with (62) and (65) yields the following,

$$\begin{aligned} \mathcal{N}_G \lesssim\ & \frac{v_{2n}^{1/2}\, M^{(n+1)/2}}{2^{n/2}\, \pi^{3/2}\, \alpha'^{1/2}\, M_p\, r_{\rm min}^{1/2}} \\ & \times \left(\frac{g_s}{\mathrm{Vol}(X_5)}\right)^{n/4} \left(\frac{K}{M}\right)^{n/4} \exp\left[\left\{ \frac{2\pi^2}{g_s^{1/2}\, \mathrm{Vol}(X_5)^{1/2}} - \frac{2\pi}{3 g_s} \left(\frac{K}{M}\right)^{1/2} \right\} \left(\frac{K}{M}\right)^{1/2} \right], \end{aligned} \tag{C4}$$

where we pulled r out of the integration by introducing the smallest $r = r_{\rm min}$ in the region of integration, thereby an upper bound for $\mathcal{N}_G$ is obtained.

The second line of (C4) is expressed as a function of K/M. It increases monotonically when K/M is small, and when

$$\frac{K}{M} = \frac{9 g_s}{8} \left\{ \frac{\pi^2}{\mathrm{Vol}(X_5)} + \frac{n}{3\pi} + \left(\frac{\pi^4}{\mathrm{Vol}(X_5)^2} + \frac{2 n \pi}{3 \mathrm{Vol}(X_5)} \right)^{1/2} \right\}, \tag{C5}$$

it reaches its maximum value

$$\left[\frac{3g_s}{4\pi}\left\{\frac{\pi^2}{\mathrm{Vol}(X_5)}+\left(\frac{\pi^4}{\mathrm{Vol}(X_5)^2}+\frac{2n\pi}{3\mathrm{Vol}(X_5)}\right)^{1/2}\right\}\right]^{n/2}\times\exp\left[\frac{3\pi^3}{4\mathrm{Vol}(X_5)}-\frac{n}{4}+\frac{3\pi}{4}\left(\frac{\pi^4}{\mathrm{Vol}(X_5)^2}+\frac{2n\pi}{3\mathrm{Vol}(X_5)}\right)^{1/2}\right], \quad \text{(C6)}$$

and then it drops sharply as K/M becomes larger than (C5). Note that the maximum value (C6) is a monotonically increasing function of g_s, and a monotonically decreasing function of $\mathrm{Vol}(X_5)$.

Assuming values such as $\mathrm{Vol}(X_5)=\pi^3$, $v_2=4\pi$, $v_4=\frac{8}{3}\pi^2$, $g_s=0.1$, and $\alpha' M_p^2=1000$, then from (C4), (C5), and (C6) the following upper bounds can be obtained:

$$\text{(D5)}\quad \mathcal{N}_G \lesssim 3\times 10^{-3}\frac{M}{r_{\min}^{1/2}}\left(\frac{K}{M}\right)^{1/4}\exp\left[\left\{11-21\left(\frac{K}{M}\right)^{1/2}\right\}\left(\frac{K}{M}\right)^{1/2}\right]\leq 7\times 10^{-3}\times\frac{M}{r_{\min}^{1/2}}, \quad \text{(C7)}$$

$$\text{(D7)}\quad \mathcal{N}_G \lesssim 8\times 10^{-4}\frac{M}{r_{\min}^{1/2}}\left(\frac{K}{M}\right)^{1/2}\exp\left[\left\{11-21\left(\frac{K}{M}\right)^{1/2}\right\}\left(\frac{K}{M}\right)^{1/2}\right]\leq 1\times 10^{-3}\times\frac{M}{r_{\min}^{1/2}}. \quad \text{(C8)}$$

The far right hand sides of (C7) and (C8) are obtained when $K/M\sim 0.1$. For example, let us substitute $N=75852$, $M=10K\simeq 870$, and $r_{\min}=0.002$, then the upper bounds for $\mathcal{N}_G$ are about 140 for a D5- and 20 for a D7-brane. However, we should remark that $r_{\min}$ could be smaller than 0.002. Perturbations at scales smaller than the CMB scale could be generated by standard slow-roll inflation. Even if DBI inflation continues until the end of inflation, if $1-n_s$ decreases at scales smaller than the CMB scale, then the lower bound for r in (36) becomes smaller than 0.002.

Since the upper bound for $\mathcal{N}_G$ varies with the parameters (especially sensitive to K/M and $\mathrm{Vol}(X_5)$), we end this appendix by stating that the region where the CMB scale is generated depends on the details of the DBI inflation model.

APPENDIX D: DETERMINANT OF $G_{kl}-B_{kl}$

Taking a gauge in which the metric G_{kl} is diagonalized, the difference between the $n\times n$ G_{kl} and an antisymmetric B_{kl} takes the following form:

$$G_{kl}-B_{kl}=\begin{pmatrix} G_{11} & B_{12} & \dots & B_{1n}\\ -B_{12} & G_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{n-1n}\\ -B_{1n} & \dots & -B_{n-1n} & G_{nn}\end{pmatrix}. \quad \text{(D1)}$$

The expansion of the determinant of (D1) in terms of the diagonal components of G_{kl} is

$$\det(G_{kl}-B_{kl})=\sum_{m=0}^{n}\sum_{G^{n-m}} G_{i_{m+1}i_{m+1}}\cdots G_{i_n i_n}\times\det\begin{pmatrix} 0 & B_{i_1i_2} & \dots & B_{i_1i_m}\\ -B_{i_1i_2} & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & B_{i_{m-1}i_m}\\ -B_{i_1i_m} & \dots & -B_{i_{m-1}i_m} & 0\end{pmatrix}, \quad \text{(D2)}$$

where $\sum_{G^{n-m}}$ requires to sum up all the combinations of choosing $(n-m)$ numbers of different G_{kk}.

The matrix on the right hand side of (D2) is built from $G_{kl}-B_{kl}$ by erasing the $i_{m+1}, i_{m+2}, \cdots, i_n$th rows and $i_{m+1}, i_{m+2}, \cdots, i_n$th columns, and changing the diagonal terms to zero. Since it is an $m\times m$ antisymmetric matrix, its determinant is zero for odd m. Hence $\det(G_{kl}-B_{kl})$ is expanded with even-ordered G_{ll}.

If m is even, the determinant of an antisymmetric matrix is equal to the square of the Pfaffian of the matrix, which is a polynomial in the components. When X is a $2p\times 2p$ antisymmetric matrix with components x_{ij}, the Pfaffian of X is

$$\mathrm{Pf}(X)=\frac{1}{2^p p!}\sum_{\sigma\in S_{2p}}\mathrm{sgn}(\sigma)\prod_{i=1}^{p}x_{\sigma(2i-1)\,\sigma(2i)}, \quad \text{(D3)}$$

where S_{2p} is the symmetric group. Then the determinant of X is

$$\det(X) = \mathrm{Pf}(X)^2. \tag{D4}$$

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